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# Weak Harnack inequality for fully nonlinear PDEs with superlinear growth terms in $Du$

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## 1 Introduction

In this note, we present the weak Harnack inequality for  $L^p$ -viscosity nonnegative supersolutions of fully nonlinear elliptic PDEs with unbounded coefficients and inhomogeneous terms. Moreover, we discuss the case when PDEs have superlinear growth terms in  $Du$ .

Throughout this paper, we suppose at least

$$p > \frac{n}{2}.$$

For measurable sets  $U \subset \mathbf{R}^n$ , we use the standard  $L^p$ -norm and  $W^{2,p}$ -norm,  $\|\cdot\|_{L^p(U)}$  and  $\|\cdot\|_{W^{2,p}(U)}$ , respectively. We will write  $\|\cdot\|_p$  and  $\|\cdot\|_{2,p}$  for them if there is no confusion. We also use the following notation:

$$L_+^p(U) = \{u \in L^p(U) \mid u \geq 0 \text{ a.e. in } U\}.$$

Let  $S^n$  be the set of  $n \times n$  symmetric matrices with the standard order. For fixed uniform ellipticity constants  $0 < \lambda \leq \Lambda$ , we denote by  $S_{\lambda,\Lambda}^n$  the set of all  $A \in S^n$  such that  $\lambda I \leq A \leq \Lambda I$ . We then define the Pucci operators  $\mathcal{P}^\pm$ : for  $X \in S^n$ ,

$$\mathcal{P}^+(X) = \max\{-\text{trace}(AX) \mid A \in S_{\lambda,\Lambda}^n\},$$

$$\mathcal{P}^-(X) = \min\{-\text{trace}(AX) \mid A \in S_{\lambda,\Lambda}^n\}.$$

Note that  $X \rightarrow \mathcal{P}^+(X)$  (resp.,  $\mathcal{P}^-(X)$ ) is convex (resp., concave).

Let us consider the most general PDEs of second-order:

$$F(x, u, Du, D^2u) = f(x) \tag{1}$$

in  $\Omega$ , where  $\Omega \subset \mathbf{R}^n$  is an open set. Here, we suppose that  $F : \Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \rightarrow \mathbf{R}$  and  $f : \Omega \rightarrow \mathbf{R}$  are given measurable functions, and that  $F$  is continuous in the last three variables.

**Definition 1.1** We call  $u \in C(\Omega)$  an  $L^p$ -viscosity subsolution (resp., supersolution) of (1) in  $\Omega$  if

$$\operatorname{ess\,lim\,inf}_{y \rightarrow x} \{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \leq 0$$

$$\left( \text{resp., } \operatorname{ess\,lim\,sup}_{y \rightarrow x} \{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \geq 0 \right)$$

whenever  $\phi \in W_{\text{loc}}^{2,p}(\Omega)$  and  $x \in \Omega$  is a local maximum (resp., minimum) point of  $u - \phi$ . Finally, we call  $u \in C(\Omega)$  an  $L^p$ -viscosity solution of (1) in  $\Omega$  if it is an  $L^p$ -viscosity subsolution and an  $L^p$ -viscosity supersolution of (1) in  $\Omega$ .

In order to memorize the right inequality, we will often say that  $u$  is an  $L^p$ -viscosity (sub)solution of

$$F(x, u, Du, D^2u) \leq f(x)$$

when it is an  $L^p$ -viscosity subsolution of (1) for instance.

We also recall the notion of strong solutions.

**Definition 1.2** We call  $u \in W_{\text{loc}}^{2,p}(\Omega)$  an  $L^p$ -strong subsolution (resp., supersolution) of (1) in  $\Omega$  if  $u$  satisfies

$$F(x, u(x), Du(x), D^2u(x)) - f(x) \leq 0 \quad (\text{resp., } \geq 0) \quad \text{a.e. in } \Omega.$$

Finally, we call  $u \in W_{\text{loc}}^{2,p}(\Omega)$  an  $L^p$ -strong solution of (1) in  $\Omega$  if the equality holds in the above.

**Remark 1.3** Suppose that  $p > p' > n/2$ . It is trivial to see that  $u$  is an  $L^p$ -strong subsolution (resp., supersolution) of (1) in  $\Omega$ , then it is an  $L^{p'}$ -strong subsolution (resp., supersolution) of (1) in  $\Omega$ . However, for  $L^p$ -viscosity solutions, the opposite implication holds true; if  $u$  is an  $L^{p'}$ -viscosity subsolution (resp., supersolution) of (1) in  $\Omega$ , then it is also an  $L^p$ -viscosity subsolution (resp., supersolution) of (1) a.e. in  $\Omega$ .

## 2 Known results

Since the weak Harnack inequality is derived from the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, we recall it from [8]. Thus, in what follows, we only consider the case when  $F$  is independent of  $u$ -variable.

Now we suppose the uniform ellipticity for  $F$ :

$$\mathcal{P}^-(X - Y) \leq F(x, \xi, X) - F(x, \xi, Y) \leq \mathcal{P}^+(X - Y)$$

for  $x \in \Omega$ ,  $\xi \in \mathbf{R}^n$ , and  $X, Y \in S^n$ . A typical example of  $F$  is given by

$$F(x, \xi, X) := \max_{1 \leq i \leq M} \min_{1 \leq j \leq N} \{-\text{trace}(A(x; i, j)X) + \langle b(x; i, j), \xi \rangle\},$$

where for  $M, N > 1$ , functions  $x \in \Omega \rightarrow A(x; i, j) \in S_{\lambda, \Lambda}^n$  and  $x \in \Omega \rightarrow b(x; i, j) \in \mathbf{R}^n$  are measurable ( $1 \leq i \leq M$ ,  $1 \leq j \leq N$ ). Notice that the above  $F$  is non-convex and non-concave in general.

Under the uniform ellipticity assumption, if  $u$  is an  $L^p$ -viscosity solution of (1) in  $\Omega$ , then it is also an  $L^p$ -viscosity solution of

$$\mathcal{P}^-(D^2u) + F(x, Du, O) \leq f(x), \text{ and } \mathcal{P}^+(D^2u) + F(x, Du, O) \geq f(x)$$

in  $\Omega$ . Therefore, for the sake of simplicity, instead of (1), we shall study the following extremal PDEs: for  $m \geq 1$ ,

$$\mathcal{P}^\pm(D^2u) \pm \mu(x)|Du|^m = \mp f(x), \quad (2)_{m, \pm}$$

where  $\mu, f$  are often supposed to be nonnegative.

We recall the ABP maximum principle for  $L^n$ -strong solutions of  $(2)_{1, -}$ .

**Proposition 2.1** (cf. [6]) There exist  $C_k = C_k(n, \lambda, \Lambda) > 0$  ( $k = 1, 2$ ) such that if  $f, \mu \in L_+^n(\Omega)$ , and  $u \in C(\overline{\Omega}) \cap W_{\text{loc}}^{2, n}(\Omega)$  is an  $L^n$ -strong subsolution of  $(2)_{1, -}$  in  $\Omega$ , then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_1 \exp(C_2 \|\mu\|_n^n) \|f\|_{L^n(\{u>0\})},$$

where  $\{u > 0\} := \{x \in \Omega \mid u(x) > 0\}$ .

**Remark 2.2** In the above statement, we can replace  $\|f\|_{L^n(\{u>0\})}$  by  $\|f\|_{L^n(\Gamma[u])}$ , where  $\Gamma[u]$  is the upper contact set of  $u$  in  $\Omega$ . See Gilbarg-Trudinger's book for the definition of  $\Gamma[u]$ .

From Proposition 2.1, it is trivial to obtain the corresponding result for  $L^p$ -strong supersolutions of  $(2)_{1,+}$  by taking  $v = -u$ .

Now, we recall an  $L^p$ -viscosity version of the ABP maximum principle. We will use a constant  $p_0 = p_0(n, \lambda, \Lambda) \in [\frac{n}{2}, n)$ , which was introduced in [4]. We note that  $p_0$  does not depend on  $\Omega$  because we only need to solve extremal PDEs in balls. See [8] (also [5]) for the details.

**Theorem 2.3** (cf. Proposition 2.8 and Theorem 2.9 in [8]) Assume that

$$q \geq p > p_0 \quad \text{and} \quad q > n \quad \text{hold.} \quad (3)$$

For  $\mu \in L_+^q(\Omega)$ , there exists  $C_3 = C_3(n, \lambda, \Lambda, \|\mu\|_q) > 0$  such that if  $f \in L_+^p(\Omega)$ , and  $u \in C(\overline{\Omega})$  is an  $L^p$ -viscosity subsolution of  $(2)_{1,-}$  in  $\Omega$ , then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_3 \|f\|_{L^p(\{u>0\})}.$$

**Remark 2.4** For more precise dependence of  $C_3$  with respect to  $\|\mu\|_q$ , we refer to [8].

We next consider the case when  $m > 1$  for  $(2)_{m,-}$ . In general, when  $m > 1$ , the ABP maximum principle for  $(2)_{m,-}$  fails even for classical solutions (see [7, 8]).

**Theorem 2.5** (Theorems 2.11 and 2.12 in [8]) Assume that (3) and

$$mq(n - p) < n(q - p) \quad (4)$$

holds. For  $m > 1$ , there exists  $\delta_1 = \delta_1(n, \lambda, \Lambda, m, p, q) > 0$  satisfying the following property: for  $\mu \in L_+^q(\Omega)$ , there is  $C_4 = C_4(n, \lambda, \Lambda, m, p, q, \|\mu\|_q) > 0$  such that if  $f \in L_+^p(\Omega)$  satisfies

$$\|f\|_p^{m-1} \|\mu\|_q < \delta_1,$$

and  $u \in C(\overline{\Omega})$  is an  $L^p$ -viscosity subsolution of  $(2)_{m,-}$  in  $\Omega$ , then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_4 \|f\|_{L^p(\{u>0\})}.$$

**Remark 2.6** We note that under (3), the relation (4) holds true when  $p \geq n$ . Thus, when  $p \geq n$ , we may choose arbitrary  $m > 1$ .

### 3 Weak Harnack inequality ( $m = 1$ )

From now on, we consider PDEs in cubes although it is possible to replace them by balls. We denote by  $Q_R$  the open cube with its center at the origin and with its length  $R > 0$ ;  $Q_R = (-\frac{R}{2}, \frac{R}{2}) \times \cdots \times (-\frac{R}{2}, \frac{R}{2})$ .

**Theorem 3.1** (Theorems 4.5 and 4.7 in [9]) Assume that (3) holds. There exists  $r = r(n, \lambda, \Lambda) > 0$  satisfying the following property: for  $\mu \in L_+^q(Q_2)$ , there is  $C_5 = C_5(n, \lambda, \Lambda, p, q, \|\mu\|_q) > 0$  such that if  $f \in L_+^p(Q_2)$  and  $u \in C(\overline{Q_2})$  is a nonnegative  $L^p$ -viscosity supersolution of  $(2)_{1,+}$  in  $Q_2$ , then it follows that

$$\left( \int_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq C_5 \left\{ \inf_{Q_1} u + \|f\|_{L^p(Q_2)} \right\}.$$

Idea of proof: We first reduce the assertion to the case when  $f \equiv 0$ . For this purpose, due to our strong solvability (Theorem 2.3 in [9]), we find an  $L^p$ -strong supersolution  $v \in C(\overline{Q_2}) \cap W_{\text{loc}}^{2,p}(Q_2)$  of

$$\mathcal{P}^-(D^2v) - \mu(x)|Dv| \geq f(x) \quad \text{in } Q_2$$

such that  $0 \leq v \leq C_6 \|f\|_p$  in  $Q_2$  for some  $C_6 = C_6(n, \lambda, \Lambda, p, \|\mu\|_q) > 0$ . Setting  $w := u + v$ , we see that  $w$  is an  $L^p$ -viscosity supersolution of  $(2)_{1,+}$  in  $Q_2$  with  $f \equiv 0$ . Thus, if we verify the assertion when  $f \equiv 0$ , then we find  $C_7 = C_7(n, \lambda, \Lambda, p, q, \|\mu\|_q) > 0$  such that

$$\left( \int_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq \left( \int_{Q_1} w^r dx \right)^{\frac{1}{r}} \leq C_7 \inf_{Q_1} w \leq C_7 \inf_{Q_1} u + C_7 C_6 \|f\|_p,$$

which concludes our proof.

Next, by considering  $U := u/(\inf_{Q_1} u + \varepsilon)$  ( $\forall \varepsilon > 0$ ), it is enough to show that  $\inf_{Q_1} u \leq 1$  implies that  $\int_{Q_1} u^r dx \leq C_0$  for some  $r, C_0 > 0$ , which are independent of  $u$  and  $\varepsilon > 0$ . (In fact, we can prove a weaker fact that  $\inf_{Q_3} u \leq 1$  implies  $\int_{Q_1} u^r dx \leq C_0$ . However, we skip this because we will not go into the details of “cube-decomposition lemma”.)

By the strong solvability (Theorem 2.3 in [8]) again, we then choose an  $L^p$ -strong solution  $\phi \in C(\overline{Q_2}) \cap W_{\text{loc}}^{2,p}(Q_2)$  of

$$\mathcal{P}^-(D^2\phi) + \mu(x)|D\phi| = \xi(x) \quad \text{in } Q_2$$

such that  $0 \geq \phi$  in  $Q_2$ ,  $-2 \geq \phi$  in  $Q_1$ , and  $\xi \in C(Q_2)$  with  $\text{supp } \xi \subset Q_1$ . Setting  $V := -u - \phi$ , we see that  $V$  is an  $L^p$ -viscosity subsolution of

$$\mathcal{P}^-(D^2V) - \mu(x)|DV| \leq -\xi(x) \quad \text{in } Q_2.$$

Hence, the ABP maximum principle (Theorem 2.3) implies

$$1 \leq \sup_{Q_1} V \leq C_3 \|\xi\|_{L^n(\{V>0\})} \leq C_3 \|\xi\|_\infty |\{x \in Q_1 \mid u(x) < M_1\}|,$$

where  $M_1 = \sup(-\phi) > 1$ . Therefore, we have

$$|\{x \in Q_1 \mid u(x) \geq M_1\}| \leq \theta$$

for some  $\theta \in (0, 1)$ . It is now enough to obtain

$$|\{x \in Q_1 \mid u(x) \geq M_1^k\}| \leq \theta^k \quad (5)$$

because this yields  $\int_{Q_1} u^r dx \leq C_0$  for some  $r, C_0 > 0$ . To prove (5), we need a “cube-decomposition” lemma (e.g. in [1, 2]) but we omit this here.

## 4 Weak Harnack inequality ( $m > 1$ )

To follow the argument in section 3, we need to establish the existence of  $L^p$ -strong solutions of the associated extremal PDEs:

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m = f(x).$$

In order to show the strong solvability of the above PDEs, we will apply the Schauder fixed point theorem. To this end, we use a recent result by Winter in [14] on the global  $W^{2,p}$ -estimate of  $L^p$ -viscosity solutions of extremal PDEs:

$$\mathcal{P}^+(D^2u) = f(x) \quad \text{in } B_1$$

under “smooth” Dirichlet condition.

Our strong solvability result is as follows:

**Theorem 4.1** (Theorem 3.1 in [10]) Assume that  $\partial\Omega \in C^{1,1}$ ,  $f \in L^p(\Omega)$ ,  $\mu \in L^q(\Omega)$  and  $\psi \in W^{2,p}(\Omega)$  hold. Assume also that one of the following conditions holds:

$$\begin{cases} (a) & q = \infty, p_0 < p, m(n-p) < n, \\ (b) & n < p \leq q < \infty, \\ (c) & p_0 < p \leq n < q < \infty, mq(n-p) < n(q-p). \end{cases} \quad (6)$$

There exists  $\delta_2 = \delta_2(n, \lambda, \Lambda, p, q, m, \Omega) > 0$  such that if

$$\|\mu\|_q (\|f\|_p + \|\psi\|_{2,p})^{m-1} < \delta_2,$$

then there exists  $L^p$ -strong solutions  $u \in W^{2,p}(\Omega)$  of

$$\begin{cases} \mathcal{P}^+(D^2u) + \mu(x)|Du|^m = f(x) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases}$$

Moreover, there is  $C_8 = C_8(n, \lambda, \Lambda, p, q, m, \Omega) > 0$  such that

$$\|u\|_{2,p} \leq C_8(\|f\|_p + \|\psi\|_{2,p}).$$

Idea of proof: It is enough to verify that we can apply the Schauder fixed point theorem to the mapping  $T : v \in W^{1,r}(\Omega) \rightarrow Tv \in W^{2,p}(\Omega)$  (for some  $r > 1$ ), where  $w := Tv$  is an  $L^p$ -strong solution of

$$\begin{cases} \mathcal{P}^+(D^2w) + \mu(x)|Dv|^m = f(x) & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases}$$

See [10] for the details.

Since we do not know if the weak Harnack inequality holds true even when  $\mu$  is bounded, we will also consider this case. We refer to [13] for related results.

**Theorem 4.2** (Theorem 4.2 in [10]) Assume that one of (6) holds. Assume also that

$$1 < m < 2 - \frac{n}{q}. \quad (7)$$

For  $M > 0$ , there exist  $\delta_3 = \delta_3(n, \lambda, \Lambda, p, m, M) > 0$ ,  $C_9 = C_9(n, \lambda, \Lambda, p, q, m) > 0$  and  $r = r(n, \lambda, \Lambda, p, q, m) > 0$  such that if  $f \in L_+^p(Q_2)$  and  $\mu \in L_+^q(Q_2)$  satisfy

$$\|\mu\|_q (1 + \|f\|_p^{m-1}) < \delta_3,$$

and an  $L^p$ -viscosity supersolution  $u \in C(Q_2)$  of  $(2)_{m,+}$  in  $Q_2$  satisfies  $0 \leq u \leq M$  in  $Q_2$ , then it follows that

$$\left( \int_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq C_9 \left\{ \inf_{Q_1} u + \|f\|_p \right\}.$$



**Remark 4.3** The hypothesis (7) is necessary when we use a scaling argument to apply the cube-decomposition lemma.

Idea of proof: In section 3, we used strong solvability of extremal PDEs  $(2)_{1,\pm}$  twice in the idea of proof of Theorem 3.1. Instead, we need to utilize Theorem 4.1 here. In order to obtain (5), we have to modify the scaling argument in [1] (also [2]) as in [11].

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